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Nonlinear extensional vibrations of quartz rods

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The one-dimensional scalar differential equation describing the extensional motion of thin piezoelectric rods is obtained from the general nonlinear three-dimensional description. Only the elastic nonlinearities are considered. The relations between the quadratic and cubic coefficients of the rod and the fundamental anisotropic elastic constants of various orders are derived. The quadratic rod coefficients are calculated for various orientations of quartz rods, but not the cubic rod coefficients because the fundamental elastic constants of fourth order, which are required for the calculation, are not presently known. The nonlinear equation and boundary conditions are applied in the analyses of both intermodulation and nonlinear resonance of quartz rods. In each instance a lumped parameter representation of the solution, which is valid in the vicinity of a resonance, is obtained and the influence of the external circuitry is included in the treatment.

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INTRODUCTION

A one-dimensional scalar nonlinear differential equation describing the extensional motion of thin piezoelectric rods oriented in an arbitrary direction with respect to the principal axes of an arbitrarily anisotropic crystal is obtained from the general nonlinear three-dimensional equations of electroelasticity.[1] Only the elastic nonlinearities are included in the description. The electrical behavior is taken to be linear since quartz has small piezoelectric coupling. The treatment provides the relation between the quadratic and cubic nonlinear extensional coefficients of the rod and the fundamental anisotropic elastic constants of second, third, and fourth order, along with the well-known relation between Young's modulus and the fundamental second-order elastic constants. The quadratic rod coefficients are calculated for various orientations of quartz rods with respect to the principal axes of the quartz crystal. Such calculations cannot be performed for the cubic rod coefficients because the fourth-order elastic constants of quartz, on which the cubic coefficients depend, are not presently known.

The steady-state solution of the nonlinear system is determined at an intermodulation frequency[2] by straightforward iteration from the linear solution. The intermodulation current–voltage relation, which is valid in the vicinity of a resonance, is obtained and the crystal is incorporated in the test circuit and the relation between the intermodulation and driving voltages is determined. This relation and the calculated quadratic rod coefficients can be used along with measurements to determine the cubic rod coefficients for any desired orientation of quartz rod, including, of course, the practical 5° *X* cut.

In addition, the nonlinear equation and boundary conditions are applied in the analysis of nonlinear resonance in quartz rods. The steady-state solution is obtained by means of an asymptotic iterative procedure[3] and an expansion in the linear eigensolutions. A lumped parameter representation of the nonlinear solution, which is valid in the vicinity of a resonance, is presented. The representation relates the amplitude of the rod displacement nonlinearly to the voltage across the electrodes. The external circuitry is incorporated in the analysis and an equation relating the amplitude of the rod displacement nonlinearly to the driving voltage and other circuit parameters is obtained.

It should be carefully noted that the coupling between extension and flexure, which can exist in an anisotropic rod, is not included in this description. However, since the coupling of a high overtone of flexure with a low extensional mode occurs only for certain well separated geometries,[4] the equation for the anisotropic rod presented here is valid for all other geometric ratios. This means that the equation is valid for almost all geometries and certainly for all practical cases.

I. NONLINEAR EQUATIONS

The stress equations of motion and charge equation of electrostatics for material points of an electroelastic solid with small piezoelectric coupling may be written in the respective forms[1]

$$K_{Lj,L} = \delta_{jM}\,\rho^0 \ddot{u}_M, \tag{1}$$

$$D_{L,L} = 0, \tag{2}$$

where we have employed the conventions that capital and lower case indices, respectively, refer to the reference Cartesian coordinates and present Cartesian coordinates of material points. We also employ the conventions that a comma followed by a capital index denotes partial differentiation with respect to the known reference coordinates, i.e., the independent variables excluding time, a dot over a variable denotes partial differentiation with respect to time, and repeated tensor indices are to be summed. The symbols ρ^0, u_M, K_{Lj}, and D_L, respectively, denote the reference mass

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density, the mechanical displacement, the Piola–Kirchhoff stress tensor, and the electric displacement vector. In the nonlinear stress equations of motion, Eq. (1), it is understood that the motion of a material point is described by the functional relation

$$y_i = y_i(X_L, t), \tag{3}$$

which is one-to-one and differentiable as often as required, where y_i and X_L, respectively, denote the present and reference position of material points. The symbol δ_{jM} in Eq. (1) is a translation operator, which serves to translate a vector from the present to the reference position and vice-versa and is required for notational consistency and clarity because of the use of capital and lower case indices, respectively, to refer to the reference and present positions of material points. Clearly, the mechanical displacement vector u_M and present and reference positions of material points y_i and X_M are related by

$$y_i = \delta_{iM}(X_M + u_M). \tag{4}$$

For purposes of this treatment, the constitutive equations for K_{Lj} and D_L may be written in the form[1]

$$K_{Lj} = y_{j,M} T_{LM}, \tag{5}$$

$$D_L = e_{LMK} E_{MK} + \epsilon^S_{LK} \mathscr{E}_K, \tag{6}$$

where

$$T_{LM} = \underset{2}{c}_{LMFG} E_{FG} - e_{KLM} \mathscr{E}_K + \tfrac{1}{2} \underset{3}{c}_{LMFGAB} E_{FG} E_{AB} + \tfrac{1}{6} \underset{4}{c}_{LMFGABCD} E_{FG} E_{AB} E_{CD}, \tag{7}$$

and

$$E_{AB} = \tfrac{1}{2}(y_{i,A}\, y_{i,B} - \delta_{AB}) \tag{8}$$

is the finite strain tensor. Substituting from Eq. (4) into Eq. (8) we obtain

$$E_{AB} = \tfrac{1}{2}(u_{A,B} + u_{B,A} + u_{K,A} u_{K,B}), \tag{9}$$

and, of course, since the electric field is quasistatic, we have[5]

$$\mathscr{E}_K = -\varphi_{,K}. \tag{10}$$

In view of the small piezoelectric coupling in quartz, in (6) and (7), we have included nonlinear elastic terms only and kept the electric and electroelastic terms linear, in accordance with the discussion in the Introduction. In Eqs. (6), (7), and (10) φ is the electric scalar potential and $\underset{2}{c}_{LMAB}$, $\underset{3}{c}_{LMKNAB}$, $\underset{4}{c}_{LMABNKIJ}$, e_{RLM}, and ϵ^S_{LK} are the second-order elastic, third-order elastic, fourth-order elastic, piezoelectric, and dielectric constants, respectively. The associated boundary conditions at the surface of the piezoelectric solid are on $N_L K_{Lj}$ or u_M and $N_L D_L$ or φ, where N_L denotes the unit normal to the undeformed surface.

A schematic diagram of the piezoelectric rod is given in Fig. 1, where $2l \gg 2w > 5h$ and the surfaces normal to x_3 are fully electroded. Since we are interested in solutions which vary slowly compared to $2h$, to lowest order the electrical variables may be written in the form[6,7]

$$\mathscr{E}_1 = \mathscr{E}_2 = 0, \tag{11a}$$

$$\mathscr{E}_3 = -V/2h, \tag{11b}$$

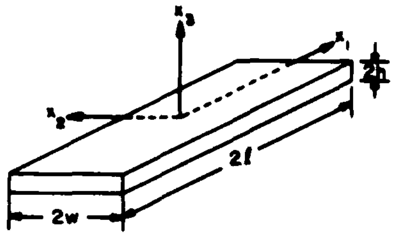

FIG. 1. Schematic diagram of the electroded piezoelectric rod.

$$D_3 = D_3(X_1, t), \tag{11c}$$

where V is the voltage across the electrodes, and D_1 and D_2 are not needed here, and to this order deviations from what is shown in Eqs. (11) are negligible. Since the electrical behavior is linear and the linear electric constitutive equation adjusted for the rod is well known,[6] it is convenient to omit the electric field $\mathscr{E}_K$ from Eq. (7) while obtaining the nonlinear rod equation and then introduce the *known* electric quantities suitably adjusted for the rod.[8]

Since we are interested in wavelengths of the order of l and, hence, solutions varying slowly compared with h (and w), both the inertia and stiffness of the electrodes are negligible and with the rod axis along X_1 we have the boundary conditions

$$K_{2j} = 0 \text{ at } X_2 = \pm w, \quad K_{3j} = 0 \text{ at } X_3 = \pm h. \tag{12}$$

Since for the low frequencies of interest the solution varies slowly compared to the cross-sectional dimensions and flexure is uncoupled from extension, which we are considering, the transverse accelerations are negligible exactly as in the linear theory of elastic rods and we have

$$\ddot{u}_2 \approx 0, \quad \ddot{u}_3 \approx 0, \tag{13}$$

while we must retain the displacement gradients $u_{2,M}$ and $u_{3,M}$ that quasistatically accompany the dynamic longitudinal deformation $u_{1,1}$. In view of Eq. (12), the aforementioned slow variation and the smallness[9] of w and h, we see that we should take

$$K_{2j} = 0, \quad K_{3j} = 0 \text{ everywhere.} \tag{14}$$

Furthermore, from Eqs. (1), (13), and (14), we have

$$K_{12,1} \approx 0, \quad K_{13,1} \approx 0, \tag{15}$$

and hence in the nonlinear description being obtained here K_{12} and K_{13} should also vanish approximately because they vanish on the surfaces at $X_1 = \pm l$. Since $y_{j,M}$ is nonsingular, from Eqs. (5) and (14) we have

$$T_{2M} = 0, \quad T_{3M} = 0, \tag{16}$$

which shows that in this description of long wavelength extensional vibrations of elastic rods all T_{LM} vanish except T_{11}. At this stage the only nonzero components of the Piola–Kirchhoff stress tensor K_{Lj} are the K_{1j}, which from Eqs. (4) and (5) are given by

$$K_{11} = (1 + u_{1,1}) T_{11}, \tag{17}$$

$$K_{12} = u_{2,1} T_{11}, \quad K_{13} = u_{3,1} T_{11}. \tag{18}$$

In the approximation being made here $u_{1,1}$ is large and all the other $u_{L,M}$, which arise because of the vanishing of the T_{LM}

($LM \neq 11$), are expressible in terms of $u_{1,1}$ and are somewhat smaller. These relations are derived in the next section and in the course of the derivation it is shown that for wavelengths large compared to the cross-sectional dimensions of the rod

$$u_{2,1} \approx 0, \quad u_{3,1} \approx 0. \tag{19}$$

Thus, from Eqs. (18) and (19) we have

$$K_{12} \approx 0, \quad K_{13} \approx 0, \tag{20}$$

and from Eqs. (14) and (20) we see that all $K_{Lj} = 0$ except K_{11}. Under these circumstances all that remains of Eq. (1) is

$$K_{11,1} = \rho^0 \ddot{u}_1, \tag{21}$$

which is the basic differential equation for the anisotropic rod. We must now find the nonlinear relation between K_{11} and $u_{1,1}$ that results from Eqs. (16). This is done in the next section.

II. NONLINEAR ANISOTROPIC ROD EQUATIONS

Since all stress components T_{LM} have been taken to vanish except T_{11}, Eq. (7) can be put in a more useful form for our purposes. To this end we first write Eq. (7) in the usual compressed matrix notation, thus

$$T_P = \underset{2}{c}_{PQ} E_Q - e_{KP} \mathscr{E}_K + \tfrac{1}{2} \underset{3}{c}_{PQR} E_Q E_R + \tfrac{1}{6} \underset{4}{c}_{PQRS} E_Q E_R E_S, \tag{22}$$

where[10]

$$P, Q, R, S, T = 1,2,\ldots,6, \tag{23}$$

and we recall from Eq. (9) that the E_Q are nonlinear in the $u_{K,L}$. We now operate on Eq. (22) with the reciprocal matrix of the second-order elastic constants c_{TP}^{-1}, which are the elastic compliances s_{TP}, to obtain

$$E_T = s_{T1} T_1 + d_{KT} \mathscr{E}_K - \tfrac{1}{2} s_{TP} \underset{3}{c}_{PQR} E_Q E_R - \tfrac{1}{6} s_{TP} \underset{4}{c}_{PQRS} E_Q E_R E_S, \tag{24}$$

where we have employed the condition that all T_P vanish except T_1 and

$$d_{KT} = e_{KP} s_{TP} \tag{25}$$

is a defined piezoelectric constant that is particularly useful when approximations are made as a result of traction-free boundary conditions. For purposes of this work it is convenient to write the six equations in Eq. (24) as two separate equations, one for $T = 1$ and the other five for $T \neq 1$, which in the absence of the piezoelectric term $d_{KT} \mathscr{E}_K$ take the forms

$$T_1 = \frac{1}{s_{11}} E_1 + \frac{1}{2} \frac{s_{1P}}{s_{11}} \underset{3}{c}_{PQR} E_Q E_R + \frac{1}{6} \frac{s_{1P}}{s_{11}} \underset{4}{c}_{PQRS} E_Q E_R E_S, \tag{26}$$

$$E_T = s_{T1} T_1 - \tfrac{1}{2} s_{TP} \underset{3}{c}_{PQR} E_Q E_R - \tfrac{1}{6} s_{TP} \underset{4}{c}_{PQRS} E_Q E_R E_S, \quad T \neq 1. \tag{27}$$

The iterative procedure consists of first solving the linear portion of Eq. (26) for ${}_0T_1$, where the zero denotes the order of the iterate, and then substituting into the linear portion of Eq. (27) to obtain ${}_0E_T$. Then the ${}_0E_T$ is substituted into the quadratic terms in Eq. (26) to obtain ${}_1T_1$, which is then substituted into Eq. (27) along with the substitution of ${}_0E_T$ into the quadratic terms to obtain ${}_1E_T$. Finally, the ${}_1E_T$ is substituted into both the quadratic and cubic terms in Eq. (26) and only quadratic and cubic terms in E_1 are retained to obtain ${}_2T_1$. Accordingly, the zeroth iterate takes the form

$${}_0T_1 = E_1/s_{11}, \tag{28a}$$

$${}_0E_T = s_{T1} E_1/s_{11}, \tag{28b}$$

which are the well-known results in the linear theory. Now, substituting Eqs. (28) into Eq. (26) and retaining terms no higher than quadratic, we obtain

$${}_1T_1 = E_1/s_{11} + (1/2s_{11}^3) s_{1P} \underset{3}{c}_{PQR} s_{1Q} s_{1R} E_1^2, \tag{29}$$

which is the first iterate for T_1 and is all we would need if we want a quadratic relation for K_{11}. Substituting from Eq. (29) and Eq. (28b) into Eq. (27) and retaining terms no higher than quadratic in E_1, we obtain

$${}_1E_T = \frac{s_{T1}}{s_{11}} E_1 + \frac{1}{2s_{11}^2} \left(\frac{s_{T1}}{s_{11}} s_{1P} - s_{TP} \right) \underset{3}{c}_{PQR} s_{Q1} s_{R1} E_1^2, \quad T \neq 1. \tag{30}$$

The further substitution of Eq. (30) into Eq. (26) and the retention of terms no higher than cubic in E_1 yields

$${}_2T_1 = cE_1 + \beta_1 E_1^2 + \gamma_1 E_1^3, \tag{31}$$

where

$$c = \frac{1}{s_{11}}, \quad \beta_1 = \frac{1}{2} \frac{s_{1P}}{s_{11}^3} \underset{3}{c}_{PQR} s_{Q1} s_{R1}, \tag{32a}$$

$$\gamma_1 = \frac{1}{2s_{11}^4} \left[s_{1P} \underset{3}{c}_{PQR} s_{R1} \left(\frac{s_{\hat{Q}1}}{s_{11}} s_{1U} - s_{\hat{Q}U} \right) \underset{3}{c}_{UVW} s_{V1} s_{W1} + \frac{1}{3} s_{1P} \underset{4}{c}_{PQRS} s_{Q1} s_{R1} s_{S1} \right], \tag{32b}$$

where the caret on the subscript Q means that

$$s_{\hat{Q}Y} = 0 \ \text{for} \ \hat{Q} = 1. \tag{33}$$

The second iterate for T_1 given in Eq. (31) with Eqs. (32) is the cubic relation we need to obtain the cubic constitutive relation for K_{11} in terms of $u_{1,1}$. Substituting from Eq. (31) with Eq. (9), and Eq. (19) into Eq. (17) and reintroducing the linear piezoelectric term, we obtain the result

$$K_{11} = cu_{1,1} + \beta (u_{1,1})^2 + \gamma (u_{1,1})^3 - (d_{31}/s_{11}) \mathscr{E}_3, \tag{34}$$

where

$$\beta = \beta_1 + \tfrac{3}{2} c, \tag{35a}$$

$$\gamma = \gamma_1 + 2\beta_1 + \tfrac{1}{2} c. \tag{35b}$$

Equation (34), with Eqs. (35) and (32), is the constitutive relation we have been seeking for the arbitrarily anisotropic piezoelectric rod, which is nonlinear in the mechanical displacement gradient $u_{1,1}$ but linear in the electric field.

We must now show that K_{12} and K_{13} do indeed vanish approximately as indicated in Sec. I. To this end we note that

at the zeroth iterate, from Eq. (28b) with Eq. (9), in tensor notation, we have

$$S_{KL} = r_{KL11}\, u_{1,1}, \tag{36}$$

where

$$S_{KL} = \tfrac{1}{2}(u_{K,L} + u_{L,K}), \tag{37a}$$

$$r_{KL11} = s_{KL11}/s_{1111}, \tag{37b}$$

and $u_1 = u_1(X_1, t)$. For $KL = 22$ and 33, respectively, we have

$$u_{2,2} = r_{2211}\, u_{1,1}, \quad u_{3,3} = r_{3311}\, u_{1,1}, \tag{38}$$

which may be integrated immediately to give

$$u_2 = X_2 r_{2211}\, u_{1,1} + f(X_1, X_3, t),$$

$$u_3 = X_3 r_{3311}\, u_{1,1} + g(X_1, X_2, t). \tag{39}$$

From the remaining three relations in Eq. (36) with Eq. (37a), we have

$$\tfrac{1}{2}(u_{1,2} + u_{2,1}) = r_{1211}\, u_{1,1}, \tag{40a}$$

$$\tfrac{1}{2}(u_{1,3} + u_{3,1}) = r_{1311}\, u_{1,1}, \tag{40b}$$

$$\tfrac{1}{2}(u_{2,3} + u_{3,2}) = r_{2311}\, u_{1,1}. \tag{40c}$$

Since the rod axis lies along X_1 and we have assumed that torsion is uncoupled from extension in this *thin* anisotropic rod, we have

$$w_{32} = \tfrac{1}{2}(u_{2,3} - u_{3,2}) = 0, \tag{41}$$

where w_{32} is the small (linear) local mechanical rotation about the rod axis X_1. From Eqs. (39), (40c), and (41), we obtain

$$f = X_3 r_{2311}\, u_{1,1}, \quad g = X_2 r_{2311}\, u_{1,1}. \tag{42}$$

Now, substituting from Eqs. (39) and (42) into (40a) and (40b), respectively, we obtain

$$\begin{aligned} u_{1,2} &= 2r_{1211}\, u_{1,1} - (X_2 r_{2211} + X_3 r_{2311}) u_{1,11}, \\ u_{1,3} &= 2r_{1311}\, u_{1,1} - (X_3 r_{3311} + X_2 r_{2311}) u_{1,11}. \end{aligned} \tag{43}$$

Since we are interested in the limit of long wavelengths (λ) as the rod collapses to a line, we have

$$\frac{\max|X_2|}{\lambda} = \frac{w}{\lambda} \to 0, \quad \frac{\max|X_3|}{\lambda} = \frac{h}{\lambda} \to 0, \tag{44}$$

which with Eqs. (39), (40c), (41), and (43) yields

$${}_0u_{1,2} = 2r_{1211}\, u_{1,1}, \tag{45a}$$

$${}_0u_{1,3} = 2r_{1311}\, u_{1,1}, \tag{45b}$$

$${}_0u_{2,1} = 0, \tag{45c}$$

$${}_0u_{2,3} = r_{2311}\, u_{1,1}, \tag{45d}$$

$${}_0u_{3,1} = 0, \tag{45e}$$

$${}_0u_{3,2} = r_{2311}\, u_{1,1}, \tag{45f}$$

which along with (38) gives all the zero-order displacement gradients. Equations (45c) and (45e) show that at the zeroth iterate Eqs. (19) hold and since ${}_1K_{1j}$ is obtained from ${}_0u_{A,B}$, we have shown that

$${}_1K_{12} \approx 0, \quad {}_1K_{13} \approx 0. \tag{46}$$

In the case of the first iterate, from Eq. (30) with Eqs. (9) and (45), in place of Eq. (36), we obtain

$$S_{KL} = r_{KL11}\, u_{1,1} + p_{KL11}(u_{1,1})^2, \tag{47}$$

where in the compressed matrix notation

$$p_{T1} = \frac{1}{2}\frac{s_{T1}}{s_{11}} + \frac{1}{2s_{11}^2}\left(\frac{s_{T1}}{s_{11}} s_{1P} - s_{TP}\right)$$

$$\times \underset{3}{c}_{PQR}\, s_{Q1}\, s_{R1} + \alpha_{T1}, \quad T \neq 1, \tag{48}$$

and each α_{T1} consists of a sum of terms, each of which contains a quadratic combination of the r_{T1} from Eqs. (45) and (9). The exact forms of the α_{T1} are cumbersome to write because of the forms in Eqs. (45) and are not needed here. From Eqs. (47) and (37b) along with the procedure followed in obtaining Eqs. (45) in the case of the zeroth iterate, it is clear that we can obtain relations similar to those in (45) if we can obtain the relation equivalent to Eq. (41) in the case of the first iterate. It is shown in the Appendix that to second order in the mechanical displacement gradients $u_{K,L}$, the local orthogonal transformation tensor R_{KL} is given by

$$R_{KL} = \delta_{KL} + \Omega_{LK}, \tag{49}$$

where

$$\Omega_{LK} = w_{LK} + \tfrac{1}{2}\, w_{LM}\, w_{MK} + \tfrac{1}{2}(S_{KM}\, w_{ML} - S_{LM}\, w_{MK}), \tag{50}$$

and

$$w_{LK} = \tfrac{1}{2}(u_{K,L} - u_{L,K}). \tag{51}$$

In the linear theory the same type of procedure shows that

$$R_{KL} = \delta_{KL} + w_{LK}. \tag{52}$$

Since in the quadratic case Ω_{LK} takes the place of w_{LK} in the linear, Ω_{23} represents the torsional rotation of the rod in the quadratic case. Consequently, in order to uncouple torsion we take

$$\Omega_{23} = 0. \tag{53}$$

Substituting from the zeroth order results in the nonlinear terms in Eq. (50) and using the limits in Eqs. (44), from Eq. (53) we obtain for the first iterate

$$w_{23} = \tfrac{1}{2}\, r_{1211} r_{1311} (u_{1,1})^2, \tag{54}$$

in place of Eq. (41) for the zeroth iterate. From the forms in Eq. (47) and Eq. (54), which take the place of Eqs. (36) and (41), respectively, it is clear that by following the procedure used earlier in the case of the zeroth iterate, we obtain expressions similar to those obtained with the zeroth iterate but containing additional terms depending on $(u_{1,1})^2$. Furthermore, when we take the limit in Eqs. (44) we obtain

$${}_1u_{2,1} = 0, \quad {}_1u_{3,1} = 0, \tag{55}$$

and since ${}_2K_{1j}$ is obtained from ${}_1u_{A,B}$, from Eq. (18) we have shown that

$${}_2K_{12} \approx 0, \quad {}_2K_{13} \approx 0. \tag{56}$$

When all stress components $T_{LK}(K_{Lj})$ vanish except $T_{11}(K_{11})$ the linear piezoelectric constitutive equation can be written in a more useful form. To this end we first write Eq. (6) in the compressed matrix notation,[10] thus

$$D_L = e_{LQ}\, E_Q + \epsilon^S_{LK}\, \mathscr{E}_K, \tag{57}$$

and we substitute from the linear portion of Eq. (24) into Eq.

(57) and employ Eq. (25) to obtain

$$D_L = d_{LP}\, T_P + \epsilon^T_{LK}\, \mathscr{E}_K, \tag{58}$$

where

$$\epsilon^T_{LK} = \epsilon^S_{LK} + e_{LQ}\, d_{KQ}. \tag{59}$$

We now substitute from the linear portion of Eq. (24) for $T = 1$ into Eq. (58) and find

$$D_L = \frac{d_{L1}}{s_{11}} S_1 + \left(\epsilon^T_{LK} - \frac{d_{L1}\, d_{K1}}{s_{11}}\right)\mathscr{E}_K, \tag{60}$$

from which, with the aid of Eqs. (11), we obtain

$$D_3 = \frac{d_{311}}{s_{1111}} u_{1,1} + \left(\epsilon^T_{33} - \frac{d^2_{311}}{s_{1111}}\right)\mathscr{E}_3, \tag{61}$$

in which we have reintroduced the tensor notation.

The substitution of Eq. (34) with Eq. (11b) into Eq. (21) yields

$$c u_{1,11} + \beta\,[(u_{1,1})^2]_{,1} + \gamma[(u_{1,1})^3]_{,1} = \rho \ddot{u}_1, \tag{62}$$

which is the differential equation of motion of the piezoelectric (anisotropic) rod. The boundary conditions consist of the specification of K_{11} or u_1 at the ends of the rod $X_1 = \pm l$. After a solution has been obtained, the current I can be calculated from the relation

$$I = 2w \int_{-l}^{l} \dot{D}_3\, dX_1, \tag{63}$$

and we recall that $\mathscr{E}_3$ in Eqs. (34) and (61) is related to the driving voltage V by Eq. (11b).

The quadratic rod coefficient β has been calculated from Eq. (35a) with Eqs. (32a) for a number of orientations of quartz rods using the second-[11] and third-[12] order elastic constants of quartz and the results are plotted in Figs. 2–4. The linear rod constant c is plotted along with $-\beta$ in each figure. The quadratic rod coefficient β is always negative and ranges in value from about $-c$ to about $-5c$. Figure 2 is for the rotated X cuts with $\theta = 0°$ corresponding to the rod axis

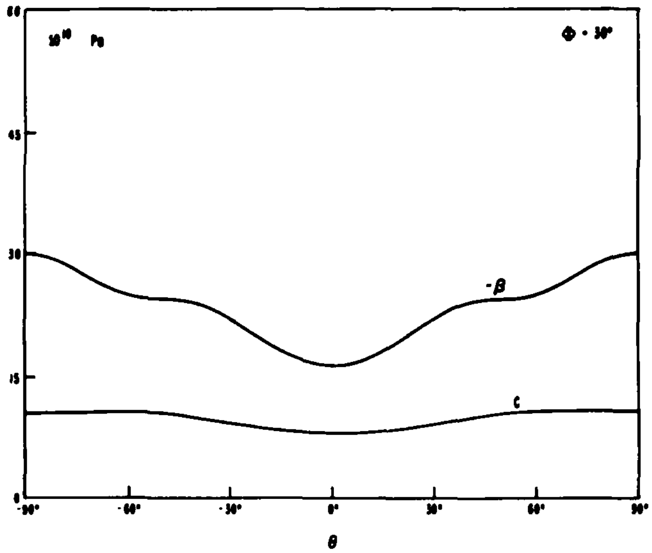

FIG. 3. Linear and quadratic elastic constants for rotated Y-cut quartz rods.

along the crystallographic Y direction. The locations of the $+5°-$, $MT-$, and $-18°-X$ cuts are shown in the figure. Figure 3 is for the rotated Y cuts with $\theta = 0°$ corresponding to the rod axis along the crystallographic X direction. This figure exhibits the twofold symmetry of quartz with X the digonal axis. Figure 4 is for doubly rotated cuts with $\theta = 0°$ corresponding to the rod in the X–Y plane at $+45°$ from the X axis. In Figs. 2–4, $\theta = \pm 90°$ corresponds to the rod axis along the crystallographic Z direction.

III. INTERMODULATION IN RODS

In this section we consider the problem of intermodulation in piezoelectric rods. For clarity we reproduce certain of the pertinent equations obtained in Secs. I and II here. The

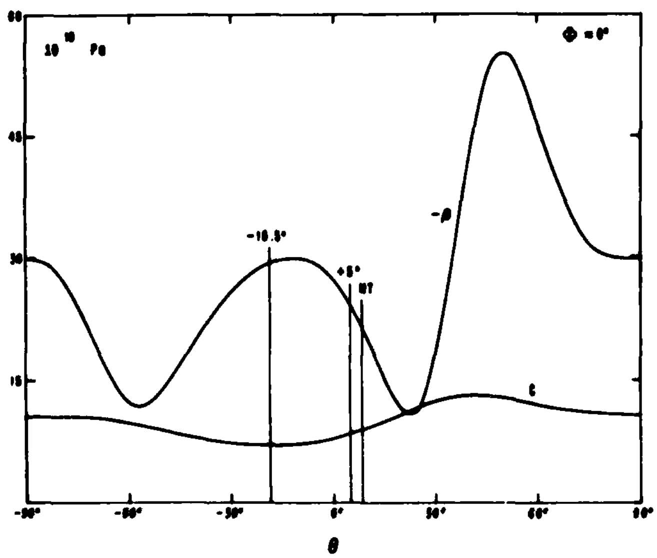

FIG. 2. Linear and quadratic elastic constants for rotated X-cut quartz rods.

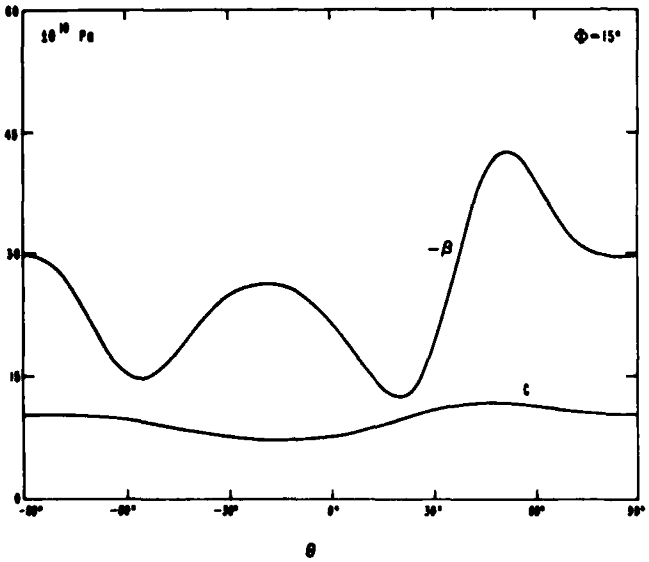

FIG. 4. Linear and quadratic elastic constants for a family of doubly rotated quartz rods.

stress equation of motion and charge equation of electrostatics for the piezoelectric rod take the form

$$K_{11,1} = \rho^0 \ddot{u}_1, \tag{64a}$$

$$D_{3,3} = 0, \tag{64b}$$

where

$$K_{11} = cu_{1,1} + \beta(u_{1,1})^2 + \gamma(u_{1,1})^3 - \frac{d_{31}}{s_{11}}\mathscr{E}_3, \tag{65}$$

$$D_3 = (d_{31}/s_{11})u_{1,1} + (\epsilon_{33}^T - d_{31}^2/s_{11})\mathscr{E}_3, \tag{66}$$

and c, β, and γ are given in Eqs. (32) and (35). From Eq. (11b) we have

$$\mathscr{E}_3 = -(V/2h)e^{i\omega t}, \tag{67}$$

where V is the magnitude of the applied voltage and ω is the steady driving frequency. For a rod free at both ends the boundary conditions are

$$K_{11} = 0 \text{ at } X_1 = \pm l. \tag{68}$$

Since the asymptotic solution to the nonlinear problem is obtained by iteration from the solution of the appropriate linear problem, we begin the analysis with a brief presentation of the known solution of the associated linear problem. As a solution of the linear problem, i.e., Eqs. (64a) and (68), with Eq. (65) in the absence of β and γ, we write

$${}_0u_1 = [{}_0u - (d_{31}/2h)VX_1]e^{i\omega t}, \tag{69}$$

the substitution of which in Eqs. (64a) and (68) yields

$$c\,{}_0u_{,11} + \rho^0\omega^2\,{}_0u = \rho^0\omega^2(d_{31}/2h)VX_1, \tag{70}$$

$${}_0u_{,1} = 0 \quad \text{at } X_1 = \pm l. \tag{71}$$

The eigensolutions of the homogeneous form of Eqs. (70) and (71), i.e., with $V = 0$, may be written in the form

$$u_{1n} = u_n e^{i\omega_n t}, \tag{72a}$$

$$u_n = a_n \sin \eta_n X_1. \tag{72b}$$

Substituting from Eqs. (72) into the homogeneous form of Eqs. (70) and (71), we obtain

$$\rho^0\omega_n^2 = c\eta_n^2, \tag{73a}$$

$$\eta_n = n\pi/2l, \quad n \text{ odd}. \tag{73b}$$

We write the steady-state solution of the forced vibration problem in the form

$${}_0u = \sum_n A_n \sin \eta_n X_1, \tag{74}$$

the substitution of which in Eq. (70) along with the use of the orthogonality of Eq. (72b) with Eq. (71) and Eq. (73a) yields

$$A_n = -8l(-1)^{(n-1)/2}d_{31}V/\{[(\omega_n^2/\omega^2) - 1]n^2\pi^2 2h\}. \tag{75}$$

Since Eq. (75) has a resonance denominator, in the vicinity of a resonance, say the Nth, one term in the series in Eq. (74) dominates and with Eq. (69) we have

$${}_0u_1 = [{}_1A_N \sin(N\pi X_1/2l) - (d_{31}V_{\omega_1}/2h)X_1]e^{i\omega_1 t}, \tag{76}$$

where we have introduced the notation ω_1, V_{ω_1}, and ${}_1A_N$ to denote the quantities at driving frequency ω_1. Substituting from Eqs. (76) and (67) at ω_1 into Eq. (66), which is then substituted into Eq. (63), we obtain

$$I_{\omega_1} = 4wli\omega_1\frac{\epsilon_{33}^T}{2h}\left(1 + \frac{8(k_{31}^l)^2}{\pi^2N^2[(\hat{\omega}_N^2/\omega_1^2) - 1]}\right)V_{\omega_1}, \tag{77}$$

where we have made the identification $I_{\omega_1} = -I$ at ω_1 since at ω_1 the crystal is passive[13] and

$$(k_{31}^l)^2 = d_{31}^2/s_{11}\epsilon_{33}^T, \tag{78}$$

and we have replaced ω_N by $\hat{\omega}_N$ where

$$\hat{\omega}_N = \omega_N + i\omega_N/2Q_N, \tag{79}$$

in which Q_N is the unloaded quality factor of the resonator in that mode, and serves to prevent the resonant denominator from vanishing at $\omega = \omega_N$.

Since we are interested in intermodulation due to the simultaneous application of test tones to the resonator at two nearby frequencies ω_1 and ω_2, both of which are in the neighborhood of ω_N, we consider the steady-state linear solution to the vibration problem already presented in this section to be at the frequencies ω_1 and ω_2. Moreover, since we are interested only in the situation in which the intermodulation frequency Ω is in the vicinity of ω_N, only certain combinations of the nonlinear frequency products are significant. To obtain expressions for the nonlinear products in Eq. (65) we first write the dominant real solution in complex notation, thus

$$u_1 = \tfrac{1}{2}({}_0u_{N1}e^{i\omega_1 t} + {}_0u_{N2}e^{i\omega_2 t} + {}_0u_{N1}^* e^{-i\omega_1 t} + {}_0u_{N2}^* e^{-i\omega_2 t}), \tag{80}$$

where * denotes complex conjugate and we have ignored the terms containing V_{ω_1} and V_{ω_2} because they do not have resonant denominators,[14] and then substitute into the quadratic product in Eq. (65) to obtain a linear steady-state inhomogeneous forced vibration problem for the first iterate solution. When the appropriate linear inhomogeneous vibration problem has been defined standard complex notation for linear systems is employed. This type of procedure, i.e., the use of the previous iterate to define the linear inhomogeneous problem for the next iterate in the above-mentioned manner, is employed at each successive stage of iteration. In view of the foregoing statements, the form of the linear solution, and previous experience[15] only the frequencies of $(2\omega_1 - \omega_2)$ and $(2\omega_2 - \omega_1)$ are of interest in our intermodulation study. Furthermore, it is sufficient for our purposes to determine the response at $(2\omega_1 - \omega_2)$ only because the response at $(2\omega_2 - \omega_1)$ can then be obtained simply by an interchange of the subscripts 1 and 2.

Since we are interested in the response at $(2\omega_1 - \omega_2)$ only, the form of Eq. (65) along with Eq. (80) and the surrounding discussion reveals that we need determine the first iterate solution ${}_1u_1$ due to the quadratic nonlinearity only at $2\omega_1$ and $(\omega_1 - \omega_2)$. Furthermore, in the interest of brevity we obtain ${}_1u_1$ at $2\omega_1$ only and simply present the changes in the result at $(\omega_1 - \omega_2)$. As already noted the first iterate equation at $2\omega_1$ is obtained by substituting ${}_0u_1$ in Eq. (76) in the quadratic term in Eq. (64a) with Eq. (65), in accordance with Eq. (80) and the associated discussion, but ignoring ${}_0u_{N2}$ and the time-independent term and neglecting the term without a resonant denominator[14] to obtain

$$c_1u_{1,11} - \rho_1^0\ddot{u}_1 = \beta_1 A_N^2\eta_N^2(\sin 2\eta_N X_1/2)e^{i2\omega_1 t}. \tag{81}$$

Since the boundary condition is given by Eq. (68), by virtue of Eqs. (73) the solution for ${}_1u_1$ takes the form

$$ {}_1u_1 = \left(\sum_m B_m \sin \eta_m X_1 - \frac{d_{31} V_{2\omega_1}}{2h} X_1\right) e^{i2\omega_1 t}, \tag{82}$$

where the B_m are obtained from the orthogonality of the $\sin \eta_m X_1$. Including only one term,[16] say the Mth, in the series in Eq. (82), we have

$$B_M = \frac{\beta_1 A_N^2 \eta_N^3 p_{NM}}{2l\rho^0(4\omega_1^2 - \hat{\omega}_M^2)} - \frac{8l(-1)^{(M-1)/2} d_{31} V_{2\omega_1}}{[(\hat{\omega}_M^2/4\omega_1^2) - 1] M^2 \pi^2 2h}, \tag{83}$$

where we have used Eq. (73a) and

$$p_{NM} = \frac{2l}{\pi}\left(\frac{\sin(2N-M)\pi/2}{2N-M} - \frac{\sin(2N+M)\pi/2}{2N+M}\right). \tag{84}$$

Substituting from Eqs. (82) and (67) at $2\omega_1$ into Eq. (66), which is then substituted into Eq. (63), we obtain

$$I_{2\omega_1} = -4wli2\omega_1 \frac{\epsilon_{33}^T}{2h}\left(1 + \frac{8(k_{31}^l)^2}{\pi^2 M^2[(\hat{\omega}_M^2/4\omega_1^2) - 1]}\right) V_{2\omega_1} + i2\omega_1 2w \frac{d_{31}}{s_{11}} \frac{\beta_1 A_N^2 \eta_N^3 p_{NM}}{l\rho^0(4\omega_1^2 - \hat{\omega}_M^2)}, \tag{85}$$

where we have made the identification $I_{2\omega_1} = I$ at $2\omega_1$ since at $2\omega_1$ the crystal is active.[13] In a similar way, again including only the Mth term,[16] at $(\omega_1 - \omega_2) = \alpha$, we obtain

$$ {}_1u_1 = \left(C_M \sin \eta_M X_1 - \frac{d_{31} V_\alpha}{2h} X_1\right) e^{i\alpha t}, \tag{86}$$

with

$$C_M = \frac{\beta_1 A_{N\,2} A_N^* \eta_N^3 p_{NM}}{2l\rho^0(\alpha^2 - \hat{\omega}_M^2)} - \frac{8l(-1)^{(M-1)/2} d_{31} V_\alpha}{[(\hat{\omega}_M^2/\alpha^2) - 1] M^2 \pi^2 2h}, \tag{87}$$

$$I_\alpha = -4wli\alpha \frac{\epsilon_{33}^T}{2h}\left(1 + \frac{8(k_{31}^l)^2}{\pi^2 M^2[(\hat{\omega}_M^2/\alpha^2) - 1]}\right) V_\alpha + i\alpha 2w \frac{d_{31}}{s_{11}} \frac{\beta_1 A_{N\,2} A_N^* \eta_N^3 p_{NM}}{l\rho^0(\alpha^2 - \hat{\omega}_M^2)}. \tag{88}$$

The linear second iterate equation at $\Omega = (2\omega_1 - \omega_2)$ is obtained by substituting the appropriate expression for ${}_1u_1$, i.e., a linear combination of the zeroth and first iterate solution functions at ω_1, ω_2, $2\omega_1$, and α, which is of the form

$$ {}_1u_1 = \frac{1}{2}\Bigg[\left({}_1A_N \sin \eta_N X_1 - \frac{d_{31} V_{\omega_1}}{2h} X_1\right) e^{i\omega_1 t} + \left({}_2A_N \sin \eta_N X_1 - \frac{d_{31} V_{\omega_2}}{2h} X_1\right) e^{i\omega_2 t} + \left(B_M \sin \eta_M X_1 - \frac{d_{31} V_{2\omega_1}}{2h} X_1\right) e^{i2\omega_1 t} + \left(C_M \sin \eta_M X_1 - \frac{d_{31} V_\alpha}{2h} X_1\right) e^{i\alpha t} + \text{c.c.}\Bigg], \tag{89}$$

in both the quadratic and cubic terms in Eq. (64a) with Eq. (65), neglecting terms without resonance denominators, i.e., terms containing V_1 and V_2 but not $V_{2\omega_1}$ and V_α, compared to terms with resonance denominators and retaining all terms of order ${}_1A_{N\,2}^2 A_N^*$ and proportional to $e^{i\Omega t}$, with the result

$$\begin{aligned} c_2 u_{1,11} - \rho^0{}_2\ddot{u}_1 = {} & (\beta/2)\big[({}_2A_N^* B_M + {}_1A_N C_M)\eta_N \eta_M \\ & \times(\eta_N \sin \eta_N X_1 \cos \eta_M X_1 + \eta_M \sin \eta_M X_1 \cos \eta_N X_1) \\ & - \eta_N^2 (d_{31}/2h)({}_2A_N^* V_{2\omega_1} + {}_1A_N V_\alpha) \sin \eta_N X_1\big] e^{i\Omega t} \\ & + \tfrac{9}{4}\gamma_1 A_{N\,2}^2 A_N^* \eta_N^4 \cos^2 \eta_N X_1 \sin \eta_N X_1 e^{i\Omega t}. \end{aligned} \tag{90}$$

As in the case of the zeroth iterate we take a series solution for ${}_2u_1$ in the form

$$ {}_2u_1 = \left(\sum_n G_n \sin \eta_n X_1 - \frac{d_{31}}{2h} V_\Omega X_1\right) e^{i\Omega t}, \tag{91}$$

and with the aid of the orthogonality of the $\sin \eta_n X_1$, we find for $n = N$, which dominates the others

$$\begin{aligned} G_N = {} & -\frac{8l(-1)^{(N-1)/2} d_{31} V_\Omega}{[(\hat{\omega}_N^2/\Omega^2) - 1] N^2 \pi^2 2h} + \frac{\beta}{2l\rho^0} \\ & \times({}_2A_N^* B_M + {}_1A_N C_M)\eta_N \eta_M\left(\eta_N q_{NM} + \frac{\eta_M p_{NM}}{2}\right) \\ & - \frac{\beta d_{31}}{4h} \frac{\eta_N^2({}_2A_N^* V_{2\omega_1} + {}_1A_N V_\alpha)}{\rho^0(\Omega^2 - \hat{\omega}_N^2)} \\ & + \frac{9}{16} \frac{\gamma_1 A_{N\,2}^2 A_N^* \eta_N^4}{\rho^0(\Omega^2 - \hat{\omega}_N^2)}, \end{aligned} \tag{92}$$

where

$$q_{NM} = \frac{\sin \eta_M l}{\eta_M} - \frac{1}{2}\left(\frac{\sin(2\eta_N + \eta_M)l}{2\eta_N + \eta_M} + \frac{\sin(2\eta_N - \eta_M)l}{2\eta_N - \eta_M}\right). \tag{93}$$

Substituting from Eqs. (91) and (67) at Ω into Eq. (66), which is then substituted into Eq. (63), we obtain

$$\begin{aligned} I_\Omega = {} & -4wli\Omega \frac{\epsilon_{33}^T}{2h}\left(1 + \frac{8(k_{31}^l)^2}{\pi^2 N^2[(\hat{\omega}_N^2/\Omega^2) - 1]}\right) V_\Omega \\ & + i\Omega\, 2w \frac{d_{31}}{s_{11}} \frac{\beta(-1)^{(N-1)/2}}{\rho^0 \hat{\omega}_N^2[(\Omega^2/\hat{\omega}_N^2) - 1]}\Bigg(({}_2A_N^* B_M \\ & + {}_1A_N C_M)\eta_N \eta_M \frac{r_{NM}}{l} - \frac{d_{31}\eta_N^2}{2h}({}_2A_N^* V_{2\omega_1} + {}_1A_N V_\alpha)\Bigg) \\ & - i\Omega\, 2w \frac{d_{31}}{s_{11}} \frac{18}{16} \frac{\gamma}{\rho^0 \hat{\omega}_N^2} \frac{{}_1A_{N\,2}^2 A_N^* \eta_N^4 (-1)^{(N-1)/2}}{[(\Omega^2/\hat{\omega}_N^2) - 1]}, \end{aligned} \tag{94}$$

where we have made the identification $I_\Omega = I$ at Ω since the crystal is active[13] at Ω and

$$r_{NM} = \eta_N q_{NM} + \eta_M p_{NM}/2. \tag{95}$$

A schematic diagram of the circuit, which is driven at the two-test tone frequencies ω_1 and ω_2, is shown in Fig. 5, where V_g is the driving voltage, R_g the generator resistance, and R_L the load resistance. Application of Kirchhoff's voltage equation to the circuit shown in Fig. 5 at ω_1 and ω_2 yields the two equations

$$\begin{aligned} {}_1V_g + I_{\omega_1}(R_g + R_L) + V_{\omega_1} &= 0, \\ {}_2V_g + I_{\omega_2}(R_g + R_L) + V_{\omega_2} &= 0, \end{aligned} \tag{96}$$

which with Eq. (77) for ω_1 and the equivalent for ω_2 enables the determination of V_{ω_1} and V_{ω_2} in terms of ${}_1V_g$ and ${}_2V_g$,

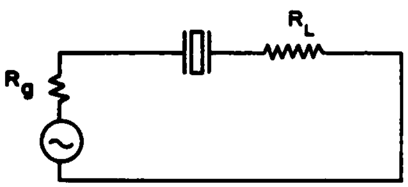

FIG. 5. Schematic diagram of the reduced test circuit.

respectively, from which ${}_1A_N$ and ${}_2A_N$ are obtained in terms of ${}_1V_g$ and ${}_2V_g$, respectively, from Eq. (75). The application of Kirchhoff's voltage equation to the circuit shown in Fig. 5 at $2\omega_1$, α, and Ω yields

$$I_\kappa(R_g + R_L) + V_\kappa = 0, \quad \kappa = 2\omega_1, \alpha, \Omega, \tag{97}$$

since at these frequencies $V_g = 0$. From Eq. (97) with $\kappa = 2\omega_1$ and $\kappa = \alpha$, respectively, and Eq. (85) and Eq. (88), $V_{2\omega_1}$ and V_α may readily be determined in terms of ${}_1V_g$ and ${}_2V_g$. Finally, from Eq. (97) with $\kappa = \Omega$ and Eq. (94), V_Ω can readily be determined in terms of ${}_1V_g$ and ${}_2V_g$ if γ is known. However, since actually γ is the only unknown quantity, a measurement of the load voltage $I_\Omega R_L$ enables the determination of I_Ω and V_Ω from Eqs. (97), from which γ can be evaluated from Eq. (94) for any desired orientation, and in particular, for the practical 5° X cut.

IV. NONLINEAR RESONANCE IN RODS

In this section we consider the problem of nonlinear resonance in piezoelectric rods. The basic nonlinear equations required for the problem of interest here are Eqs. (64)–(68). We again obtain the solution by iteration from the linear solution, but now the successive iterates must be corrected because we are interested in the steady-state nonlinear solution at the driving frequency ω, in which there is a nonlinear dependence of the resonant frequency ω_N on the amplitude of the extensional displacement of the rod. Clearly, the zeroth iterate, or linear solution, is given in Sec. III and Eqs. (69)–(79) hold for purposes of this section. Again, in the vicinity of the Nth resonance one term dominates the series and the zeroth iterate is given by Eq. (76) with the subscript 1 stricken from the A_N and the ω.

In accordance with the discussion surrounding Eq. (80) in order to obtain the linear equation for the first iterate, the zeroth iterate solution is written in the form

$$\begin{aligned} {}_0u_1 = \frac{1}{2}\bigg[&\left(A_N \sin\frac{N\pi}{2l}X_1 - \frac{d_{31}V_\omega}{2h}X_1\right)e^{i\omega t} \\ &+ \left(A_N^* \sin\frac{N\pi}{2l}X_1 - \frac{d_{31}V_\omega^*}{2h}X_1\right)e^{-i\omega t}\bigg], \end{aligned} \tag{98}$$

which is substituted in the quadratic term in Eq. (64a) with Eq. (65) to obtain the linear inhomogeneous equation for the first iterate

$$c_1u_{1,11} - \rho_1^0\ddot{u}_1 = (\beta/2)\eta_N^3 \sin 2\eta_N X_1(A_N^2e^{i2\omega t} + A_NA_N^*). \tag{99}$$

Since the boundary condition is given by Eq. (68) with Eq. (65) and the V_ω^2 is negligible, by virtue of Eq. (73b) the solution for ${}_1u_1$ takes the form

$$\begin{aligned} {}_1u_1 = &\left(\sum_m B_m \sin\eta_m X_1 - \frac{d_{31}V_{2\omega}}{2h}\right)e^{i2\omega t} \\ &+ \sum_m C_m \sin\eta_m X_1, \end{aligned} \tag{100}$$

where the B_m and C_m are obtained from the orthogonality of the $\sin\eta_m X_1$ and we have made use of the fact that the electrodes are shorted at zero frequency. Since the case of primary interest is $N = 1$ and for that case $M = 1$ gives the largest term in each series in Eq. (100) and also results in the largest single term caused by the series in Eq. (100) at the next iterate and we are not interested here in an exhaustive treatment of all possibilities, but only in indicating the nature of the influence of the terms in the series in Eq. (100) on the nonlinear resonance equation and obtaining the expression due to the term of greatest influence for the case $N = 1$, we consider the term $M = N$ only in each series in Eq. (100). Accordingly, including only the Mth term in both series in Eq. (100), with the aid of the orthogonality of the $\sin\eta_m X_1$ and Eq. (73a), we have

$$\begin{aligned} B_M &= \frac{\beta A_N^2\eta_N^3 p_{NM}}{2l\rho^0(4\omega^2 - \omega_M^2)} - \frac{(-1)^{(M-1)/2}8ld_{31}V_{2\omega}}{[(\omega_M^2/4\omega^2) - 1]M^2\pi^2 2h}, \\ C_M &= -\frac{\beta A_N^2\eta_N^3 p_{NM}}{2l\rho_0\omega_M^2}, \end{aligned} \tag{101}$$

where p_{NM} is given in Eq. (84). Substituting from Eq. (100) and Eq. (67) at 2ω into Eq. (66), which is then substituted into Eq. (63), we obtain

$$\begin{aligned} I_{2\omega} = &-4wli2\omega\frac{\epsilon_{33}^T}{2h}\left(1 + \frac{8(k_{31}^l)^2}{[(\omega_M^2/4\omega^2) - 1]M^2\pi^2}\right)V_{2\omega} \\ &+ i2\omega 2w\frac{d_{31}}{s_{11}}\frac{\beta A_N^2\eta_N^3 p_{NM}}{l\rho^0(4\omega^2 - \omega_M^2)}, \end{aligned} \tag{102}$$

where we have made the identification $I_{2\omega} = I$ at 2ω, since at 2ω the crystal is active.[13] In accordance with earlier discussions in order to obtain the linear equation for the second iterate, the first iterate solution is written in the form

$$\begin{aligned} {}_1u_1 = \frac{1}{2}\bigg[&\left(A_N \sin\frac{N\pi}{2l}X_1 - \frac{d_{31}V_\omega}{2h}X_1\right)e^{i\omega t} \\ &+ \left(B_M \sin\frac{M\pi}{2l}X_1 - \frac{d_{31}V_{2\omega}}{2h}X_1\right)e^{i2\omega t} \\ &+ C_M \sin\frac{M\pi}{2l}X_1 + \text{c.c.}\bigg], \end{aligned} \tag{103}$$

and substituted in both the quadratic and cubic terms in Eqs. (64a) and (68), with Eq. (65), while retaining all terms of order $A_N^2A_N^*$ and containing the time dependence $e^{i\omega t}$ only to obtain the homogeneous differential equation

$$\begin{aligned} c_2u_{1,11} - \rho^0\,{}_2\ddot{u}_1 = \frac{\beta}{2}\bigg(&(A_N^*B_M + A_NC_M)\eta_N\eta_M \\ &\times(\eta_N \sin\eta_N X_1 \cos\eta_M X_1 \\ &+ \eta_M \sin\eta_M X_1 \cos\eta_N X_1) \\ &- \eta_N^2A_N^*\frac{d_{31}V_{2\omega}}{2h}\sin\eta_N X_1\bigg)e^{i\omega t} \\ &+ \frac{9}{4}\gamma A_N^2A_N^*\eta_N^4 \cos^2\eta_N X_1 \sin\eta_N X_1 e^{i\omega t}, \end{aligned} \tag{104}$$

and the inhomogeneous boundary conditions

$$
\begin{aligned}
c_2 u_{1,1} - \frac{d_{31}}{s_{11}}\frac{V_\omega}{2h}e^{i\omega t}
&= -\frac{\beta}{2}\Big[A_N^* \eta_N \cos\eta_N l \\
&\quad \times\Big(B_M \eta_M \cos\eta_M l - \frac{d_{31}}{2h}V_{2\omega}\Big) \\
&\quad + A_N \eta_N \cos\eta_N l C_M \eta_M \cos\eta_M l\Big]e^{i\omega t} \\
&\quad - \frac{3}{4}\gamma A_N^2 A_N^* \eta_N^3 \cos^3\eta_N l e^{i\omega t}, \quad \text{at } X_1 = \pm l,
\end{aligned}
\tag{105}
$$

where we have neglected terms without resonant denominators, i.e., those containing V_ω, compared to terms with resonant denominators and we have not yet set $M = N$. It should be noted that the right-hand side of Eq. (105) turns out to be negligible because A_N is small and η_N is nearly given by Eq. (73b). Since we regard the A_N, η_N, and η_M on the right-hand side of Eqs. (104) and (105) as presently unknown and ignore Eqs. (73b) and (75), the equations are homogeneous for $V_\omega = 0$ and we say the iterate is corrected.[17]

Before we obtain the nonlinear eigensolutions of Eqs. (104) and (105) we first note that for the short circuit case $V_{2\omega} = 0$, and the open circuit case $I_{2\omega} = 0$ yields the greatest difference in eigenfrequency from the shorted case. Furthermore, the open circuit voltage $V_{2\omega}^0$ obtained from Eqs. (102) when substituted in Eq. (104) contains the product of $(k_{31}^l)^2$ and $A_N^2 A_N^*$, both of which are small. Since the product of two small quantities is negligible, we may omit $V_{2\omega}$ in Eqs. (104) and (105) in the sequel. The nonlinear eigensolutions must satisfy Eq. (104), which for $M = N$ takes the form

$$
\begin{aligned}
c_2 u_{1,11} - \rho_2^0 \ddot{u}_1 = \Big(&H \sin 2\eta_N X_1 \\
&+ \frac{9}{16}\gamma A_N^2 A_N^* \eta_N^4 (\sin\eta_N X_1 \\
&+ \sin 3\eta_N X_1)\Big)e^{i\omega t},
\end{aligned}
\tag{106}
$$

where

$$
\begin{aligned}
H &= -\frac{1}{3}\frac{\beta^2}{\gamma c}\gamma A_N^2 A_N^* \frac{f_N}{l}\eta_N^3, \\
f_N &= \Big((-1)^{(N-1)/2} - \frac{(-1)^{(N+1)/2}}{3}\Big),
\end{aligned}
\tag{107}
$$

and we have made use of the fact that we are obtaining the steady-state solution for ω in the vicinity of ω_N only and we have employed Eqs. (73), (84), and (101). The homogeneous boundary conditions are given by Eq. (105) with $V_\omega = 0$. As the Nth eigensolution of the linear differential equation (106), we take

$$
{}_2u_1 = u_N e^{i\tilde{\omega}_N t},
\tag{108}
$$

where

$$
u_N = A_N \sin\eta_N X_1 + G_N \sin 2\eta_N X_1 + L_N \sin 3\eta_N X_1,
\tag{109}
$$

the substitution of which in Eq. (106) yields

$$
\rho^0 \tilde{\omega}_N^2 = c\eta_N^2(1+\mu),
\tag{110}
$$

$$
G_N = \tfrac{32}{81}\kappa(f_N/N\pi)\mu A_N,
\tag{111a}
$$

$$
L_N = -\mu A_N/8,
\tag{111b}
$$

where

$$
\kappa = \beta^2/c\gamma,
\tag{112a}
$$

$$
\mu = \tfrac{9}{16}(\gamma/c)\eta_N^2 A_N A_N^*.
\tag{112b}
$$

Since the amplitude A_N is very small, Eqs. (111) with Eq. (112b) show that G_N and L_N are negligible for our purposes except insofar that they might result in a change in $\tilde{\omega}_N$ by virtue of the boundary conditions (105). Substituting from Eqs. (108) and (109) into Eq. (105), we obtain

$$
\begin{aligned}
\cos\eta_N l = -\mu\Big(&\cos\eta_N l + \frac{64}{81}\kappa\frac{f_N}{N\pi}\cos 2\eta_N l \\
&- \frac{8}{27}\kappa\frac{f_N}{N\pi}(1+\cos 2\eta_N l) \\
&- \frac{1}{24}\cos 3\eta_N l\Big).
\end{aligned}
\tag{113}
$$

The roots of Eq. (113), with Eq. (110), determine the amplitude dependent eigenfrequencies $\tilde{\omega}_N$ of this piezoelectric rod in the nonlinear case. In the absence of the nonlinear term μ, Eq. (113) reduces to Eq. (73b). Since μ is small, the roots $\eta_N l$ may be obtained iteratively from the roots in the linear case, which are given in Eq. (73b). Consequently, in the nonlinear case the roots must differ from $N\pi/2$ by small quantities, say Δ_N, and we may write

$$
\eta_N l = N\pi/2 + \Delta_N, \quad N \text{ odd}.
\tag{114}
$$

Substituting from Eq. (114) into Eq. (113), expanding and retaining terms linear in Δ_N, we obtain

$$
\Delta_N = -(-1)^{(N-1)/2}\tfrac{64}{81}\kappa(f_N/N\pi)\mu,
\tag{115}
$$

which shows that as a consequence of the existence of the coefficient β of the quadratic term, the nonlinear boundary conditions have a small influence[18] on η_N. Since $\mu \ll 1$, from Eq. (110), with the aid of Eqs. (114) and (115), we have

$$
\begin{aligned}
\tilde{\omega}_N = \frac{N\pi}{2l}\Big(\frac{c}{\rho^0}\Big)^{1/2}\Big(&1 + \frac{\mu}{2} \\
&- (-1)^{(N-1)/2}\frac{128}{81}\frac{\kappa f_N}{N^2\pi^2}\mu\Big),
\end{aligned}
\tag{116}
$$

which gives the resonant frequency of the Nth mode of extensional vibration of the thin piezoelectric rod with small piezoelectric coupling, including the dependence on the amplitude of vibration.[19]

The solution to the nonlinear forced vibration problem at a driving frequency ω in the vicinity of ω_N may now be obtained by means of an expansion in the eigensolutions, while retaining the nonlinear correction in the dominant Nth eigensolution only. Accordingly, we write

$$
{}_2u_1 = [{}_2u - (d_{31}V_\omega/2h)X_1]e^{i\omega t},
\tag{117}
$$

where

$$
\begin{aligned}
{}_2u = &A_N \sin\eta_N X_1 + G_N \sin 2\eta_N X_1 + L_N \sin 3\eta_N X_1 \\
&+ \sum_{n\neq N} A_n \sin\eta_n X_1, \quad n \text{ odd},
\end{aligned}
\tag{118}
$$

and G_N and L_N are determined from Eqs. (111). In the iterative procedure we are employing the influence of the large

term in the expansion in Eq. (118), i.e., $A_N \sin \eta_N X_1$, already appears on the right-hand side of Eqs. (104) and (105) including that resulting from β and the first iterate solution. Substituting from Eq. (117) with Eq. (118) into Eq. (104), we obtain

$$[-c\eta_N^2(1+\mu)+\rho^0\omega^2]A_N \sin \eta_N X_1 + \left((-c4\eta_N^2+\rho^0\omega^2)G_N + c\frac{16}{27}\kappa\frac{f_N}{l}\mu A_N\, \eta_N\right)\sin 2\eta_N X_1$$

$$+ [(-c9\eta_N^2+\rho^0\omega^2)L_N - \mu c\eta_N^2 A_N]\sin 3\eta_N X_1 + \sum_{n\neq N}(-c\eta_n^2+\rho^0\omega^2)A_n \sin \eta_n X_1 = \rho^0\omega^2\frac{d_{31}V_\omega}{2h}X_1, \tag{119}$$

and we note that the solution in Eq. (117) with Eq. (118) satisfies Eq. (105) identically because the linear terms satisfy the linear homogeneous boundary conditions (71) termwise and the nonlinear term satisfies the nonlinear transcendental equation (113) Utilizing the fact that ω is in the vicinity of ω_N, substituting from Eq. (111) into Eq. (119) and neglecting terms of order μ^2, we have

$$[-c\eta_N^2(1+\mu)+\rho^0\omega^2]A_N \sin \eta_N X_1 + \sum_{n\neq N}(-c\eta_n^2+\rho^0\omega^2)A_n \sin \eta_n X_1 = \rho^0\omega^2\frac{d_{31}V_\omega}{2h}X_1, \tag{120}$$

from which, with the aid of the orthogonality of the $\sin n\pi X_1/2l$, we obtain

$$(\omega^2-\bar{\omega}_N^2)A_N\, l = \omega^2\frac{d_{31}V_\omega 8l^2}{2hN^2\pi^2}(-1)^{(N-1)/2}, \tag{121}$$

and we do not bother to obtain any other A_n because, as already noted, we are interested in the solution only when ω is in the vicinity of ω_N and the Nth eigenmode is dominant so that all other eigenmodes are negligible. Substituting from Eqs. (116) and (112b) into Eq. (121), we obtain

$$N^2\pi^2A_N[\omega^2-\omega_N^2(1+\alpha A_N A_N^*)] = 4(-1)^{(N-1)/2}d_{31}(l/h)V_\omega\omega^2, \tag{122}$$

where ω_N is given in Eqs. (73) and

$$\alpha = \frac{9}{16}\frac{\gamma}{c}\frac{N^2\pi^2}{4l^2} - (-1)^{(N-1)/2}\frac{4}{9}\frac{\beta^2}{c^2}\frac{f_N}{l^2}. \tag{123}$$

Equation (122) gives the nonlinear relation between the amplitude of the extensional displacement of the rod and the voltage across the crystal. Thus for an ω in the vicinity of ω_N, the solution can very accurately be written

$$u_1 = \left(A_N \sin\frac{N\pi}{2l}X_1 - \frac{d_{31}V_\omega}{2h}X_1\right)e^{i\omega t},$$

$$\mathscr{E}_3 = -\frac{V_\omega}{2h}e^{i\omega t}, \tag{124}$$

where A_N must satisfy Eq. (122). Equations (122) and (124) constitute the solution of the steady-state nonlinear forced extensional vibration problem for the rod for driving frequencies in the vicinity of ω_N.

Since we wish to obtain the nonlinear relation between the amplitude A_N of the extensional motion of the rod and the driving voltage V_g shown in the circuit in Fig. 5, we substitute from Eqs. (124) into Eq. (66), which is then substituted into Eq. (63) to obtain

$$I_\omega = i\omega 4w\frac{l\epsilon_{33}^T}{2h}\left(V_\omega - \frac{2h}{l}\frac{d_{31}}{\epsilon_{33}^T s_{11}}\hat{A}_N\right), \tag{125}$$

where we have made the identification $I_\omega = -I$ since the rod is passive[13] and

$$\hat{A}_N = A_N(-1)^{(N-1)/2}. \tag{126}$$

Application of Kirchhoff's voltage equation to the circuit shown in Fig. 5 yields

$$V_g + I_\omega(R_g+R_L) + V_\omega = 0, \tag{127}$$

where V_g is the generator voltage, R_g and R_L are the generator and load resistance, respectively, and V_ω is the voltage across the crystal. Substituting from Eqs. (125) and (127) into Eq. (122), we obtain

$$\frac{N^2\pi^2}{4\omega^2}[\omega^2-\hat{\omega}_N^2(1+\alpha\hat{A}_N\hat{A}_N^*)]\left(1+i\omega\frac{2wl\epsilon_{33}^T}{h}(R_g+R_L)\right) - i\omega(R_g+R_L)4w\frac{l}{h}\epsilon_{33}^T(k_{31}^l)^2\hat{A}_N = -\frac{d_{31}l}{h}V_g, \tag{128}$$

and, as usual, we have replaced ω_N by $\hat{\omega}_N$ where

$$\hat{\omega}_N = \omega_N + i\omega_N/2Q_N, \tag{129}$$

in which Q_N is the unloaded quality factor of the resonator in the Nth mode. Equation (128) is the nonlinear equation relating the amplitude of the extensional displacement to the generator voltage. Since Eq. (128) contains $\hat{A}_N^*$ as well as $\hat{A}_N$, it is not simply a complex cubic equation. In order to facilitate the solution of Eq. (128), we multiply Eq. (128) by its complex conjugate and after some algebraic manipulation obtain

$$\mathscr{a}[d^2+C^2-2(d+Ca)\hat{\omega}_N^2\,\alpha\mathscr{a}+(1+a^2)\hat{\omega}_N^4\,\alpha^2\mathscr{a}^2] = e^2\mathscr{v}_g, \tag{130}$$

where

$$\mathscr{a} = \hat{A}_N\hat{A}_N^*, \quad \mathscr{v}_g = V_gV_g^*, \tag{131}$$

$$a = \omega(R_g+R_L)(2wl\epsilon_{33}^T/h), \quad d = \omega^2-\hat{\omega}_N^2+\hat{\omega}_N^2\,a/Q_N,$$
$$C = -(\hat{\omega}_N^2/Q_N)+a(\omega^2-\hat{\omega}_N^2)-b, \quad e = d_{31}l/h\nu,$$
$$\nu = N^2\pi^2/4\omega^2, \quad b = \omega(R_g+R_L)(4wl\epsilon_{33}^T/h)(k_{31}^l)^2/\nu. \tag{132}$$

Equation (130) is a real cubic algebraic equation in the square of the extensional vibration amplitude $\hat{A}_N\hat{A}_N^*$, which may readily be solved for the $\hat{A}_N\hat{A}_N^*$. For a solution $\hat{A}_N\hat{A}_N^*$ to be meaningful, it must be real and positive. Consequently, there is at least one and possibly three physically meaningful $\hat{A}_N\hat{A}_N^*$. When a physically meaningful $\hat{A}_N\hat{A}_N^*$ has been determined from Eq. (130), it may be substituted in Eq. (128), which then becomes a linear equation in $\hat{A}_N$ that may readily be solved for the phase if desired. When an $\hat{A}_N$ is known, the current I_ω through the circuit may be evaluated from Eq.

(125). This result can be used as a check for the γ determined from intermodulation measurements for any desired orientation and in particular, for the practical 5° X cut of quartz.

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APPENDIX

In this Appendix we derive the expression in Eq. (49) with Eq. (50) for the local orthogonal transformation containing terms up to quadratic in the mechanical displacement gradients. To this end we first note that Green's deformation tensor is defined by[20]

$$C_{KL} = y_{i,K}\, y_{i,L}, \tag{A1}$$

and that the polar decomposition theorem takes the form[21]

$$y_{i,L} = R_{iK}\, U_{KL}, \tag{A2}$$

where

$$U_{KM}\, U_{ML} = C_{KL}, \tag{A3}$$

and, of course, if U_{KL}^{-1} is the reciprocal of U_{KL}, we have

$$R_{iK} = y_{i,L}\, U_{LK}^{-1}. \tag{A4}$$

We now substitute from Eqs. (4) and (37a) into Eq. (A1) to obtain

$$C_{KL} = \delta_{KL} + 2S_{KL} + u_{i,K}\, u_{i,L}. \tag{A5}$$

Since the magnitude of $u_{K,L}$ is small, on account of Eqs. (A3) and (A5) U_{KM} must differ from the identity δ_{KM} by a small (but nonlinear) quantity γ_{KM}, thus

$$U_{KM} = \delta_{KM} + \gamma_{KM}. \tag{A6}$$

Substituting from Eq. (A6) into Eq. (A3) and employing Eq. (A5), we obtain

$$\gamma_{KL} = S_{KL} + \tfrac{1}{2} u_{i,K}\, u_{i,L} - \tfrac{1}{2}\gamma_{KM}\,\gamma_{ML}. \tag{A7}$$

Also, since $u_{K,L}$ is small, U_{ML}^{-1} must differ from δ_{ML} by a small (but nonlinear) quantity α_{ML}, thus

$$U_{ML}^{-1} = \delta_{ML} + \alpha_{ML}. \tag{A8}$$

Since

$$U_{KM}\, U_{ML}^{-1} = \delta_{KL}, \tag{A9}$$

the substitution of Eqs. (A6) and (A8) in Eq. (A9) yields

$$\alpha_{KL} = -\gamma_{KL} - \gamma_{KM}\,\alpha_{ML}. \tag{A10}$$

Furthermore, from Eq. (A4), with Eqs. (4) and (A8), we have

$$R_{iK} = \delta_{iM}\, R_{MK}, \tag{A11}$$

where

$$R_{MK} = \delta_{MK} + u_{M,K} + \alpha_{MK} + u_{M,L}\,\alpha_{LK}, \tag{A12}$$

and we have employed the translation operator δ_{iM} discussed in Sec. I in obtaining Eqs. (A11) with (A12). Equation (A12) shows that if we can find α_{LM} to any order in the small quantities $u_{K,L}$, we can obtain R_{MK} and, of course, R_{iK} to that order in $u_{K,L}$.

To lowest order, from Eqs. (A7) and (A10), successively, we obtain

$${}_0\gamma_{KL} = S_{KL}, \tag{A13a}$$

$${}_0\alpha_{KL} = -S_{KL}, \tag{A13b}$$

the substitution of the latter of which in Eq. (A12) yields the usual linear relation given in Eq. (52). To second order, using (A13) in the nonlinear terms in Eqs. (A7) and (A10), we obtain

$${}_1\gamma_{KL} = S_{KL} + \tfrac{1}{2} u_{M,K}\, u_{M,L} - \tfrac{1}{2} S_{KM}\, S_{ML}, \tag{A14a}$$

$${}_1\alpha_{KL} = -S_{KL} - \tfrac{1}{2} u_{M,K}\, u_{M,L} + \tfrac{3}{2} S_{KM}\, S_{ML}. \tag{A14b}$$

Substituting from Eq. (A13b) into the nonlinear terms and (A14b) into the linear terms in (A12) and employing Eqs. (37a) and (51), we obtain

$$R_{MK} = \delta_{MK} + \Omega_{KM}, \tag{A15}$$

where

$$\Omega_{KM} = w_{KM} + \tfrac{1}{2} w_{KL}\, w_{LM} + \tfrac{1}{2}(S_{ML}\, w_{LK} - S_{KL}\, w_{LM}), \tag{A16}$$

which, respectively, correspond to Eqs. (49) and (50). It is a simple matter to show that the orthogonality relations

$$R_{LK}\, R_{LM} = \delta_{KM}, \quad R_{LK}\, R_{MK} = \delta_{LM} \tag{A17}$$

are satisfied to second order in the $u_{K,M}$.

[1] H. F. Tiersten, "Nonlinear Electroelastic Equations Cubic in the Small Field Variables," J. Acoust. Soc. Am. **57**, 660 (1975).
[2] H. F. Tiersten, "Analysis of Intermodulation in Thickness-Shear and Trapped Energy Resonators," J. Acoust. Soc. Am. **57**, 667 (1975).
[3] H. F. Tiersten, "Analysis of Nonlinear Resonance in Thickness-Shear and Trapped Energy Resonators," J. Acoust. Soc. Am. **59**, 866 (1976).
[4] T. R. Meeker, "Extension, Flexure and Shear Modes in Rotated X-Cut Quartz Rectangular Bars," *Proceedings of the 33rd Annual Symposium on Frequency Control* (Fort Monmouth, NJ, 1979), p. 286.
[5] H. F. Tiersten, *Linear Piezoelectric Plate Vibrations* (Plenum, New York, 1969), Chap. 4, Sec. 4.
[6] *IEEE Standard on Piezoelectricity-IEEE Std. 176-1978* (Institute of Electrical and Electronics Engineers, New York, 1978), Sec. 4.5.
[7] Reference 5, Chap. 13.
[8] Since only linear electric terms are retained, this is the consistent approach.
[9] In this work we are interested in obtaining the extensional equation for the rod essentially in the limit as the cross-section collapses to a line.
[10] Reference 5, Chap. 7, Sec. 1.
[11] R. Bechmann, "Elastic and Piezoelectric Constants of Alpha-Quartz," Phys. Rev. **110**, 1060 (1958).
[12] R. N. Thurston, H. J. McSkimin, and P. Andreatch, Jr., "Third-Order Elastic Coefficients of Quartz," J. Appl. Phys. **37**, 267 (1965).
[13] Reference 5, Chap. 15, Sec. 1.
[14] Only terms with resonant denominators for driving frequencies in the vicinity of the resonant frequencies are large and need be retained in the nonlinear products.
[15] Reference 2, Sec. II. These are the only frequency combinations occurring at the lowest orders of nonlinearity that are in the vicinity of ω_N.
[16] Only one term is retained because only the largest term is considered to be of any potential importance. If more are thought to be required, they may be included without difficulty.
[17] The procedure is a straightforward iterative procedure except when the functional form resulting from the nonlinear terms is identical with that satisfying the basic linear differential equation, at which point the coefficients associated with the functional form due to the nonlinearity are treated as unknown and identical with the coefficients of the linear solu-

tion. This is the reason Eq. (104) is a homogeneous equation whereas Eq. (99) is an inhomogeneous equation. This procedure is appropriate because the term on the rhs of (104) actually is a homogeneous term and appears to be inhomogeneous (known) only because of the nature of the formal procedure employed which tends to be misleading whenever this type of coincidence of functions occurs.

[18]A similar effect was inadvertently neglected in Ref. 3 and should have been included. However, the quadratic coefficient in Ref. 3 for the AT-cut turns out to be so small that the effect has a negligible influence on the numerical results.

[19]It should be recalled that the last term in Eq. (116) is the large one for the case $N = 1$ only. For other values of N, values of $M \neq N$ have to be considered, perhaps more than one. However, if for a particular orientation κ turns out to be very small, the last term in Eq. (116) is negligible and the solution holds for any odd N.

[20]C. Truesdell and R. A. Toupin, "The Classical Field Theories," in *Encyclopedia of Physics*, edited by S. Flügge (Springer-Verlag, Berlin, 1960), Vol. III/1, Eq. (26.2).

[21]Reference 20, Eq. (37.8). The Appendix does no more than carry out to second order what the reference states can be done.